Lagrangian Relaxation in MIP

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Outline

Generic MIP model

$$Z(M) = \min cx + fy$$
$$Ax = b$$
$$Bx + Dy \geq e$$
$$Gy \geq h$$
$$x, y \geq 0$$
$$y \text{ integer}$$

- Hypotheses: M is feasible and bounded

LP relaxation

$$Z(LP(M)) = \min cx + fy$$
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- How to solve $LP(M)$?

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- How to solve $LP(M)$?
- (Primal) Simplex method
- Other methods?

Branch-and-bound (B&B) algorithm

- Lower (dual) bounds: $LP(M)$ with dual (primal) simplex
- Upper (primal) bounds: LP-based heuristics
- Pruning (fathoming): what are the rules to prune a node?

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- Branching: how does it work?
- Selection (search): depth-first or best-first or other rules?
- Other ingredients: preprocessing, reduced-cost domain reduction

Column generation method

- Assume that there are “too many” variables in x
- Generate a small subset of these (just enough to define a basic feasible solution to $LP(M)$)
- Solve the LP relaxation to obtain an optimal solution $(\bar{x}, \bar{y})$
- Solve the **pricing problem**:
  1) determine that no more x variables have a negative reduced cost
  2) otherwise, find at least one x variable with a negative reduced cost
- Case 1): stop the algorithm
  Case 2): add the columns to the model
- Repeat this iterative process until no more variables are found
- This method solves $LP(M)$, but not M! What should we do?

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- Branch-and-**price**: column generation at every node of the B&B tree

Polyhedral theory: basic definitions

- **Polyhedron**: set of solutions to a finite set of linear inequalities
- **Affine independence**: $x_0, x_1, \dots, x_k$ are affinely independent iff $\omega_i = 0$, $i = 0, 1, \dots k$, is the unique solution of the system $\sum_{i=0}^{k} \omega_i x_i = 0$, $\sum_{i=0}^{k} \omega_i = 0$
- **Dimension of polyhedron** P: maximum number of affinely independent points in a polyhedron - 1, noted $dim(P)$
- **Valid inequality**: an inequality is valid for a set S if it is satisfied by all points in the set
- **Face of polyhedron** P: a face is the set of points in P that satisfies a valid inequality for P at equality
- **Representation of face** F: the valid inequality for polyhedron P that is satisfied at equality by F is said to represent face F
- **Proper face of polyhedron** P: any face F of P such that $F \neq \emptyset, P$ (which is itself a polyhedron with $dim(F) < dim(P)$)

Facet description of polyhedra

- **Description of polyhedron** P: any set of linear inequalities whose solutions define P (there is an infinite number of descriptions for any polyhedron!)
- **Facet of polyhedron** P: proper face F of P such that $dim(F) = dim(P) - 1$
- **Description of polyhedron** P **by its facets**:
  - If F is a facet of P, then any description of P must include at least one inequality representing F (called **facet-defining inequality**)
  - Every inequality that represents a face that is not a facet is not necessary in the description of P

Extreme point description of polyhedra

- Hypothesis: all variables used to describe polyhedron P are nonnegative
- **Extreme point of polyhedron** P: a point of P that cannot be written as a convex combination of two other points of P
- Do you know any other characterization of an extreme point?

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- Do you know any other characterization of an extreme point?
- **Cone** P^0 **associated to polyhedron** P: obtained from the description of P by replacing the right-hand side by 0
- **Ray of** P: vector belonging to P^0, the cone associated to P
- **Extreme ray of** P: ray of P that cannot be written as nonnegative combination of two other rays that do not point in the same direction
- **Minkowski's theorem**: every point of P can be written as a convex combination of the extreme points of P plus a nonnegative combination of the extreme rays of P
- Make sure you understand this important result! Write it down in mathematical form. What is happening if P is a bounded polyhedron?

Convexity and polyhedra

- **Convex hull of set** S: set of points that can be expressed as convex combination of the points of S, denoted $conv(S)$
- $conv(S) = S$, if S is convex, so $conv(P) = P$ if P is a polyhedron!
- If S is the feasible region of the MIP: $\min_{x \in S} cx$, then $conv(S)$ is a polyhedron
- This implies that, by replacing the feasible region S by $conv(S)$, we obtain the LP: $\min_{x \in conv(S)} cx$

Relationships between LP and MIP

- Let $\overline{S}$ be the feasible region of the LP relaxation: $\min_{x \in \overline{S}} cx \leq \min_{x \in S} cx$
- Since $conv(S)$ is contained in any convex set that includes S, we have $conv(S) \subseteq \overline{S}$, therefore $\min_{x \in \overline{S}} cx \leq \min_{x \in conv(S)} cx$
- We can even show that, if the MIP is feasible and bounded, then $\min_{x \in S} cx = \min_{x \in conv(S)} cx$
- So, a MIP is just an LP! What is the problem, then?

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- We can even show that, if the MIP is feasible and bounded, then $\min_{x \in S} cx = \min_{x \in conv(S)} cx$
- So, a MIP is just an LP! What is the problem, then?
- In general, a description of $conv(S)$ is difficult to obtain
- One notable exception: when $conv(S) = \overline{S}$; we then say that the MIP has the **integrality property**
- This condition is verified when all extreme points of $\overline{S}$ are integral
- Do you know a class of MIP models having the integrality property?

Lagrangian relaxation

- $Bx + Dy \geq e$ can be seen as "complicating" constraints
- Relax them, but instead of dropping them, add them to the objective by associating **Lagrange multipliers** $\alpha \geq 0$ representing penalties associated to their violation

$$Z(LR_\alpha(M)) = \min cx + fy + \alpha(e - Bx - Dy)$$

$$Ax = b$$

$$Gy \geq h$$

$$x, y \geq 0$$

$$y \text{ integer}$$

- For any $\alpha \geq 0$, we have $Z(LR_\alpha(M)) \leq Z(M)$
- Prove this property!

Lagrangian subproblem

$$Z(LR_\alpha(M)) = \alpha e + Z_\alpha^x + Z_\alpha^y$$

$$Z_\alpha^x = \min(c - \alpha B)x$$
$$Ax = b$$
$$x \geq 0$$

$$Z_\alpha^y = \min(f - \alpha D)y$$
$$Gy \geq h$$
$$y \geq 0$$
$$y \text{ integer}$$

- Hypothesis: both problems are bounded

Lagrangian dual

- The best values for the Lagrange multipliers are obtained by solving the **Lagrangian dual**:

$$Z(LD(M)) = \max_{\alpha \geq 0} Z(LR_\alpha(M))$$

- Let $Y = \{y \geq 0 \text{ and integer} | Gy \geq h\}$ and recall that $conv(Y)$ is a polyhedron
- **Primal interpretation of Lagrangian duality**:

$$Z(LD(M)) = \min cx + fy$$

$$Ax = b$$

$$x \geq 0$$

$$Bx + Dy \geq e$$

$$y \in conv(Y)$$

- Show this result by using Minkowski's theorem and LP duality

Lagrangian dual and LP relaxation

- We know $conv(Y) \subseteq \overline{Y}$, where $\overline{Y} = \{y \geq 0 | Gy \geq h\}$
- The primal interpretation of Lagrangian duality implies $Z(LD(M)) \geq Z(LP(M))$
- If the Lagrangian subproblem has the integrality property ($conv(Y) = \overline{Y}$), then $Z(LD(M)) = Z(LP(M))$
- Even when this is the case, maybe there is another Lagrangian relaxation such that $Z(LD(M)) \geq Z(LP(M))$
- Write down the Lagrangian subproblem and the Lagrangian dual if one relaxes the constraints $Ax = b$

Solving the Lagrangian dual by column generation

- Solve the **Dantzig-Wolfe reformulation (master problem)** derived from the primal interpretation of Lagrangian duality and Minkowski's theorem:

$$Z(LD(M)) = \min cx + \sum_{q \in Q} \lambda_q (f y_q)$$

$$Ax = b$$

$$Bx + \sum_{q \in Q} \lambda_q (D y_q) \geq e$$

$$\sum_{q \in Q} \lambda_q = 1$$

$$x \geq 0, \; \lambda_q \geq 0, \; q \in Q$$

- Use column generation for this LP with "too many" variables
- Show that the pricing problem is the Lagrangian subproblem

Solving the Lagrangian dual by subgradient optimization

- $Z(LR_\alpha(M))$ is continuous and concave (nice), but non-differentiable (ugly) function of α
- A **subgradient** of $Z(LR(\alpha))$ at $\overline{\alpha}$ is given by $(e - B\overline{x} - D\overline{y})$, where $(\overline{x}, \overline{y})$ solves the Lagrangian subproblem for $\alpha = \overline{\alpha}$
- At each iteration, a subgradient method finds a new α by taking a step in the direction of a subgradient
- Although there are convergence results, this is essentially a heuristic method that is very quick at each iteration
- Suggestion: use subgradient optimization at the beginning to get "good" Lagrange multipliers, then switch to column generation to get the best possible lower bound

Capacitated facility location problem (CFLP)

- K: set of customers
- J: set of locations for potential facilities
- $d_k > 0$: demand of customer k
- $u_j > 0$: capacity at location j
- $f_j \geq 0$: fixed cost for opening facility at location j
- $c_{jk} \geq 0$: unit cost of satisfying the demand of customer k from facility at location j
- **Problem description**: Determine the locations of the facilities to satisfy customers' demands at minimum cost, while respecting the capacity at each facility location

Weak CFLP model

- y_j: 1, if location j is chosen for a facility, 0, otherwise
- x_{jk}: fraction of the demand d_k of customer k satisfied from facility at location j

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$$Z(W) = \min \sum_{j \in J} \sum_{k \in K} d_k c_{jk} x_{jk} + \sum_{j \in J} f_j y_j$$

$$\sum_{j \in J} x_{jk} = 1, \quad k \in K \quad (\pi_k)$$

$$\sum_{k \in K} d_k x_{jk} \leq u_j y_j, \quad j \in J \quad (\alpha_j \geq 0)$$

$$x_{jk} \in [0, 1], \quad j \in J, k \in K$$

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Weak LP relaxation and valid inequalities

- How would you solve the LP relaxation? What do you think of the lower bound $Z(LP(W))$?

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- Are they valid inequalities? Cuts? Facets of the convex hull?

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- Suggest constraints that can be added to improve the LP relaxation

$$x_{jk} \leq y_j, \quad j \in J, k \in K$$

- Are they valid inequalities? Cuts? Facets of the convex hull?
- When these inequalities are added, there is an optimal solution to the resulting **strong LP relaxation** that satisfies

$$y_j = \max\left\{\max_{k \in K}\{x_{jk}\}, \frac{\sum_{k \in K} d_k x_{jk}}{u_j}\right\}$$

Lagrangian relaxation of capacity constraints

$$Z(LR_\alpha(W)) = \min \sum_{j \in J} \sum_{k \in K} d_k(c_{jk} + \alpha_j) x_{jk} + \sum_{j \in J} (f_j - \alpha_j u_j) y_j$$

$$\sum_{j \in J} x_{jk} = 1, \quad k \in K$$

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- Does it have the integrality property?

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$$\max_{\alpha \geq 0} Z(LR_\alpha(W)) \equiv Z(LD^C(W)) = Z(LP(W))$$

Lagrangian relaxation of assignment constraints

$$Z(LR_\pi(W)) = \min \sum_{j\in J} \sum_{k\in K} (d_k c_{jk} - \pi_k) x_{jk} + \sum_{j\in J} f_j y_j + \sum_{k\in K} \pi_k$$

$$\sum_{k\in K} d_k x_{jk} \leq u_j y_j, \quad j \in J$$

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$$\max_\pi Z(LR_\pi(W)) \equiv Z(LD^A(W)) \ge Z(LP(W))$$

Strong LP relaxation

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$$\sum_{k \in K} d_k x_{jk} \leq u_j y_j, \quad j \in J \quad (\alpha_j \geq 0)$$

$$x_{jk} \leq y_j, \quad j \in J, k \in K \quad (\beta_{jk} \geq 0)$$

$$x_{jk} \geq 0, \quad j \in J, k \in K$$

$$y_j \in \{0, 1\}, \quad j \in J$$

Lagrangian relaxation of capacity constraints

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Lagrangian relaxation of capacity and linking constraints

$$Z(LR_{(\alpha,\beta)}(S)) = \min \sum_{j\in J} \sum_{k\in K} (d_k c_{jk} + d_k \alpha_j + \beta_{jk}) x_{jk}$$

$$+ \sum_{j\in J} (f_j - \alpha_j u_j - \sum_{k\in K} \beta_{jk}) y_j$$

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Some polyhedral results

- The last Lagrangian subproblem has the integrality property:

$$conv\{(x_j, y_j) \geq 0 | \sum_{k \in K} d_k x_{jk} \leq u_j y_j;\ x_{jk} \leq y_j,\ k \in K;\ y_j \in \{0, 1\}\}$$

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- This immediately implies the following result:

$$Z(LD^A(W)) = Z(LD^A(S))$$

Improving the Lagrangian bound

- Show that the following inequality is valid:

$$\sum_{j \in J} u_j y_j \geq \sum_{k \in K} d_k$$

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- Does this inequality improve the strong LP relaxation bound?

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$$\sum_{j \in J} u_j y_j \geq \sum_{k \in K} d_k$$

- Does this inequality improve the strong LP relaxation bound?
- Adding this inequality to the model (defining S^+) seems useless: $Z(LP(S)) = Z(LP(S^+))$!
- This is true in particular if we add this inequality and then relax it in a Lagrangian way: why?

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- Adding this inequality to the model (defining S^+) seems useless: $Z(LP(S)) = Z(LP(S^+))$!
- This is true in particular if we add this inequality and then relax it in a Lagrangian way: why?
- But, what about adding it, but also keeping it in the Lagrangian subproblem?

Lagrangian relaxation of capacity and linking constraints

$$Z(LR_{(\alpha,\beta)}(S)) = \min \sum_{j\in J}\sum_{k\in K}(d_k c_{jk} + d_k\alpha_j + \beta_{jk})x_{jk}$$

$$+ \sum_{j\in J}(f_j - \alpha_j u_j - \sum_{k\in K}\beta_{jk})y_j$$

$$\sum_{j\in J} x_{jk} = 1, \quad k \in K$$

$$x_{jk} \geq 0, \quad j \in J, k \in K$$

$$\sum_{j\in J} u_j y_j \geq \sum_{k\in K} d_k$$

$$y_j \in \{0,1\}, \quad j \in J$$

- How would you solve this problem?

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$$
\max_{(\alpha,\beta)\geq 0} Z(LR_{(\alpha,\beta)}(S^+)) \equiv Z(LD^{CL}(S^+)) \geq Z(LP(S^+))
$$

Lagrangian relaxation of assignment constraints

$$Z(LR_\pi(S^+)) = \min \sum_{j \in J} \sum_{k \in K} (d_k c_{jk} - \pi_k) x_{jk} + \sum_{j \in J} f_j y_j + \sum_{k \in K} \pi_k$$

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- How would you solve this problem?
- Does it have the integrality property?

$$\max_\pi Z(LR_\pi(S^+)) \equiv Z(LD^A(S^+)) \geq Z(LP(S^+))$$

Comparison of lower bounds

$$
\begin{aligned}
Z(W) = Z(S) = Z(S^{+}) &\geq Z(LD^{A}(S^{+})) \\
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&\geq Z(LP(S^{+})) \\
&= Z(LD^{A}(S)) \\
&= Z(LD^{CL}(S)) \\
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Note: no dominance between $Z(LD^A(S^+))$ and $Z(LD^C(S))$

Lagrangian heuristic with subgradient optimization

- Lower bound $Z(LD^A(S^+))$: computed approximately with subgradient optimization
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$(\lvert J\rvert, \lvert K\rvert)$	$Z(LP(S))$	$Z(LD^A(S^+))$	$Z(LH)$	$Z(S^+)$
(100,1000)	2.12, 111	0.06, 9	0.00, 9	0.00, 7145
(100,1000)	2.92, 50	0.05, 7	0.00, 7	0.00, 7004
(100,1000)	1.81, 19	0.20, 3	0.00, 3	0.00, 19
(100,1000)	0.00, 3	*0.01*, 1	0.00, 1	0.00, 3
(100,1000)	0.37, 53	0.01, 2	0.00, 2	0.00, 53
(100,1000)	0.40, 37	0.30, 5	2.16, 5	0.00, 37
(100,1000)	0.33, 29	0.33, 5	1.36, 5	0.00, 29
(100,1000)	0.09, 18	*0.12*, 9	0.00, 9	0.00, 18
(100,1000)	0.53, 19	0.16, 8	0.00, 8	0.00, 19
(100,1000)	0.42, 12	*0.44*, 6	0.39, 6	0.00, 12
(100,1000)	0.09, 5	0.09, 5	0.42, 5	0.00, 5
(100,1000)	0.05, 3	0.05, 6	0.03, 6	0.00, 3

Dantzig-Wolfe reformulation

- Reminder: the Lagrangian dual can be solved with column generation on the **Dantzig-Wolfe reformulation**
- Write down Dantzig-Wolfe reformulation for $LD^A(S^+)$

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- Using this fact, write down an equivalent **disaggregated** Dantzig-Wolfe reformulation

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- Write down Dantzig-Wolfe reformulation for $LD^A(S^+)$
- Reminder: the Lagrangian subproblem decomposes into $|J|$ continuous knapsack problems linked with a 0-1 knapsack problem
- Using this fact, write down an equivalent **disaggregated** Dantzig-Wolfe reformulation
- Q: set of extreme points of the convex hull of the 0-1 knapsack problem
- R_j: set of extreme points of the convex hull of the continuous knapsack problem for $j \in J$

Disaggregated Dantzig-Wolfe reformulation

$$Z(LD^A(S^+)) = \min \sum_{j \in J} \sum_{r \in R_j} \theta_j^r \left(\sum_{k \in K} d_k c_{jk} x_{jk}^r \right) + \sum_{q \in Q} \lambda^q \left(\sum_{j \in J} f_j y_j^q \right)$$

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$$\lambda^q \geq 0, \quad q \in Q$$

$$\theta_j^r \geq 0, \quad j \in J, r \in R_j$$

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- Reduced cost of variable λ^q:

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- What is the pricing problem?
- Define $\omega_j(\pi) \geq 0$ as the opposite of the optimal value of the continuous knapsack problem for $j \in J$
- If $\sum_{j \in J} (f_j - \omega_j(\pi)) y_j^q - \delta < 0$, variable λ^q is added, along with variables θ_j^r such that $\omega_j(\pi) > \omega_j$
- If $\sum_{j \in J} (f_j - \omega_j(\pi)) y_j^q - \delta \geq 0$ and $\omega_j(\pi) \leq \omega_j$ for each $j \in J$, the column generation method has converged to $Z(LD^A(S^+))$

Branch-and-price algorithm

- By adding the constraint $\lambda^q \in \{0, 1\}$, $q \in Q$, we obtain a reformulation of the CFLP
- At each node, we perform the column generation method
- What happens in the pricing problem if we branch on λ^q?

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- At each node, we perform the column generation method
- What happens in the pricing problem if we branch on λ^q?
- When we branch, we have to make sure that we do not destroy the structure of the pricing problem!
- A better way to branch is:
  - Choose j such that $\sum_{q \in Q} \lambda^q y_j^q$ is fractional
  - Generate child nodes defined by the branching constraints:
    1) $\sum_{q \in Q} \lambda^q y_j^q = 0$
    2) $\sum_{q \in Q} \lambda^q y_j^q = 1$
- Show that this branching rule is valid and does not change the way we solve the pricing problem