INTRODUCTION TO DYNAMIC DISCRETE CHOICE MODELS: FROM STATIC TO DYNAMIC

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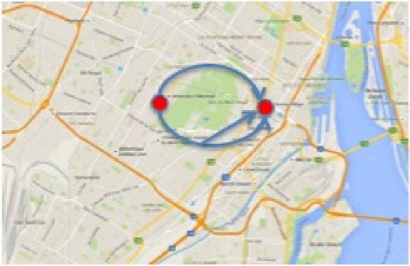
OUTLINE

- Introduction to route choice modelling
- Recursive logit
- Rust's model
- Maximum likelihood estimation

INTRODUCTION

- Route choice models play an important role in different transport related applications (e.g. traffic simulation and evaluation of infrastructure investments)



INTRODUCTION

- Choice set of three paths $C_n = \{1,2,3\}$
- Travel time is static and deterministic and is the only attribute observed by the modeller
- $V_1 = V_2 = V_3 = V_j = \beta T_j$
- Static logit model $U_j = V_j + \varepsilon_j$

$$P(1|C_n) = P(2|C_n) = P(3|C_n) = \frac{e^{V_1}}{\sum_{j \in C_n} e^{V_j}}$$

INTRODUCTION

- Transport networks are large and there are many alternatives for each origin-destination pair
- Several challenges associated with route choice modelling: choice sets are unknown and path utilities are correlated
- Addressing the choice set issue: we can reformulate the path choice problem as a sequence of link choices, we call it recursive logit (Fosgerau et al., 2013, Mai et al. 2015)
- Recursive logit for a static and deterministic network is a simple case of a dynamic discrete choice model

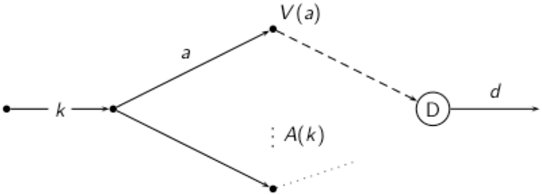
RECURSIVE LOGIT

- Utilities are associated with links $a \in A$ and the deterministic part is $-T_a$, here -4 (upper link) and -2 for the others
- Logit over outgoing links at origin $\frac{e^{-2}}{e^{-2}+e^{-4}} = 0.88$
- Recursive logit $\frac{e^{-2-0.57}}{e^{-2-0.57}+e^{-4}} = 0.50$

RECURSIVE LOGIT

- Directed connected network $G = (A, \mathcal{V})$
- Deterministic attributes
- A state $k \in A$ is a link in the network so that turn attributes can be included
- Choice of one outgoing link (action) $a \in A(k)$ at the sink node of k
- Path: $(k_0, .., k_I)$ with $k_{i+1} \in A(k_i)$ for all $i < I$
- From now on
  - All equations are destination specific
  - Notation for individuals omitted
  - Static network is assumed

RECURSIVE LOGIT



- Infinite horizon problem with an absorbing state d (destination)
- $\widetilde{A} = A \cup d$
- $u(a|k) = v(a|k) + \mu\varepsilon(a)$, $\varepsilon(a)$ i.i.d. EV type 1
- $v(a|k) = v(x_{a|k};\beta) < 0 \ \forall a,k \in A$, $v(d|k) = 0$

RECURSIVE LOGIT

- Traveler chooses next link given current state, stochastic process having Markov property
- **Next state given with certainty by the action**
- Traveler observes $\varepsilon(a)\ \forall a \in A(k)$, chooses action a that maximizes sum of $u(a|k)$ and expected maximum downstream utility $V(a)$

RECURSIVE LOGIT

- Bellman equations

$$V(k) = E\left[\max_{a\in A(k)} \left(v(a|k) + V(a) + \mu\varepsilon(a)\right)\right] \;\forall\; k \in A$$

- Infinite horizon but no discounting!

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RECURSIVE LOGIT

- Probability of choosing link a given state k

$$P(a|k) = \frac{e^{\frac{1}{\mu}(v(a|k)+V(a))}}{\sum_{a' \in A(k)} e^{\frac{1}{\mu}(v(a'|k)+V(a'))}}$$

RECURSIVE LOGIT

- Expected maximum utility corresponds to the "logsum"

$$V(k) = \begin{cases} \mu \ln \sum_{a \in A} \delta(a|k) e^{\frac{1}{\mu}(v(a|k)+V(a))} & \forall k \in A \\ 0 & k = d \end{cases}$$

$\delta(a|k) = 1$ if $a \in A(k)$ and zero otherwise

- Note that denominator in $P(a|k)$ is $e^{\frac{1}{\mu}V(k)}$

PROPERTIES

- Solving the Bellman equations

$$e^{\frac{1}{\mu}V(k)} = \begin{cases} \sum_{a\in A} \delta(a|k) e^{\frac{1}{\mu}(v(a|k)+V(a))} & \forall\ k \in A, \\ 1 & k = d \end{cases}$$

PROPERTIES

- System of linear equations

$$\mathbf{z} = \mathbf{M}\mathbf{z} + \mathbf{b} \Leftrightarrow (\mathbf{I} - \mathbf{M})\mathbf{z} = \mathbf{b}$$

- $\mathbf{z}$ $(|\widetilde{A}| \times 1)$, $z_k = e^{\frac{1}{\mu}V(k)}$
- $\mathbf{b}$ $(|\widetilde{A}| \times 1)$, $b_k = 0\ \forall k \in A$, $b_k = 1, k = d$
- $\mathbf{M}$ $(|\widetilde{A}| \times |\widetilde{A}|)$

$$M_{ka} = \begin{cases} \delta(a|k)e^{\frac{1}{\mu}v(a|k)} & \forall a \in \widetilde{A},\ \forall k \in A \\ 0 & \text{otherwise} \end{cases}$$

PROPERTIES

- The system has a solution if $(\mathbf{I}-\mathbf{M})$ is invertible which depends on the balance between the number of paths that connect nodes in the network and the size of $\frac{1}{\mu}v(a|k)$
- $\frac{1}{\mu}v(a|k)$ are network dependent since they are scaled to the variance of the error terms
- Dense networks and many alternative paths do not necessarily imply that $\mathbf{I}-\mathbf{M}$ is ill-conditioned

PROPERTIES

- Path probabilities
- Observation $\sigma = \{k_i\}_{i=0}^{I}$, k_0 is the origin and $k_I = d$
- By the Markov property $P(\sigma) = \prod_{i=0}^{I-1} P(k_{i+1}|k_i)$

$$
\begin{aligned}
P(\sigma) &= \prod_{i=0}^{I-1} e^{\frac{1}{\mu}(v(k_{i+1}|k_i)+V(k_{i+1})-V(k_i))} \\
&= e^{-\frac{1}{\mu}V(k_0)} \prod_{i=0}^{I-1} e^{\frac{1}{\mu}v(k_{i+1}|k_i)}
\end{aligned}
$$

- Denote $v(\sigma) = \sum_{i=0}^{I-1} v(k_{i+1}|k_i)$

PROPERTIES

$$P(\sigma) = \frac{e^{\frac{1}{\mu}v(\sigma)}}{e^{\frac{1}{\mu}V(k_0)}} = \frac{e^{\frac{1}{\mu}v(\sigma)}}{\sum_{\sigma' \in \Omega} e^{\frac{1}{\mu}v(\sigma')}}$$

- Path based multinomial logit model with an infinite number of alternatives
- Ω set of all paths (including those with loops)
- IIA property holds over paths

ESTIMATION

- A sample of observations of (different) travellers making path choices σ_n, $n = 1, \ldots, N$, e.g. collected by GPS
- Maximum likelihood estimation

$$\max_{\beta} LL_N(\beta) = \frac{1}{N} \sum_{n=1}^{N} \ln P(\sigma_n; \beta) \qquad (1)$$

ESTIMATION

- Log-likelihood function

$$LL(\beta) = \ln \prod_{n=1}^{N} P(\sigma_n;\beta) = \frac{1}{\mu} \sum_{n=1}^{N} \sum_{t=0}^{T_n-1} v(k_{t+1}|k_t;\beta) - V(k_0;\beta)$$

- Unlike a static logit model we need to solve the value functions in order to evaluate choice probabilities
- A non-linear optimization algorithm is used to search over the parameter space
- We use the NFXP algorithm since the value functions are not expensive to compute

APPLICATION

- Borlänge network (3077 nodes, 7459 links, 21452 link pairs)
- The value functions can be solved using a direct solver (MATLAB)
- Validation using synthetic data
- Estimation based on real observations

APPLICATION – VALIDATION

- Instantaneous utilities

$$v(a|k) = \beta_{TT}TT_a + \beta_{LT}LT_{a|k} + \beta_{LC}LC_a + \beta_{UT}UT_{a|k}$$

- Travel time in minutes (TT)
- Left turn dummy (LT) for turns larger than 40 degrees
- Link constant (LC) to penalize paths with many crossings
- U-turn dummy (UT) to remove u-turns from the network, fixed to -20

APPLICATION – VALIDATION

- One origin-destination pair in Borlänge
- 10 samples of 500 paths observations
- Chosen parameters: $\beta_{TT} = -2$, $\beta_{LT} = -1$, $\beta_{LC} = -1$ and $\beta_{UT} = -20$

APPLICATION – VALIDATION

Sample	$\widehat{\beta}_{TT}$	Std. Err.	$\widehat{\beta}_{LT}$	Std. Err.	$\widehat{\beta}_{LC}$	Std. Err.
1	-1.91	0.21	-1.02	0.09	-1.07	0.06
2	-1.97	0.22	-0.99	0.09	-1.04	0.06
3	-1.80	0.21	-1.09	0.09	-1.07	0.06
4	-2.38	0.26	-0.88	0.09	-1.01	0.06
5	-2.20	0.24	-0.96	0.08	-0.93	0.05
6	-2.30	0.26	-0.96	0.09	-0.96	0.06
7	-1.69	0.18	-1.00	0.08	-1.11	0.06
8	-1.84	0.20	-1.04	0.08	-1.04	0.05
9	-2.40	0.29	-1.05	0.09	-0.89	0.06
10	-1.88	0.20	-0.99	0.08	-0.976	0.05
Average	-2.04	0.23	-1.00	0.09	-1.01	0.06
Std. Err.	0.26		0.06		0.07	

APPLICATION – REAL DATA

- Real data: 1832 path observations, 466 destinations and over 37000 link choices
- Comparison with path based logit models

Parameters	RL-LS	RL	PSL	PL
$\widehat{\beta}_{TT}$	-3.060	-2.494	-2.738	-2.431
Rob. Std. Err.	0.103	0.098	0.086	0.083
Rob. t-test (0)	-27.709	-25.449	-31.837	-29.289
$\widehat{\beta}_{LT}$	-1.057	-0.933	-1.000	-0.920
Rob. Std. Err.	0.029	0.030	0.027	0.029
Rob. t-test (0)	-36.448	-31.100	-37.037	-31.724
$\widehat{\beta}_{LC}$	-0.353	-0.411	-0.545	-0.429
Rob. Std. Err.	0.011	0.013	0.012	0.013
Rob. t-test (0)	-32.091	-31.615	-45.417	-33.000
$\widehat{\beta}_{UT}$	-4.531	-4.459	-4.366	-4.375
Rob. Std. Err.	0.126	0.114	0.118	0.119
Rob. t-test (0)	-35.960	-39.114	-37.000	-36.765
$\widehat{\beta}_{LS}$ and $\widehat{\beta}_{PS}$	-0.227		1.461	
Rob. Std. Err.	0.013		0.082	
Rob. t-test (0)	-17.462		17.817	
$\widehat{LL}_N(\widehat{\beta})$	-3.300	-3.441	-1.601	-1.688

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CROSS VALIDATION

- Cross validation approach: 80% for estimation, 20% for prediction
- 40 holdout samples of the same size taken from the real sample
- Log-likelihood loss, averaged over holdout samples

$$\overline{err}_p = \frac{1}{p}\sum_{i=1}^{p} err_i \quad \forall 1 \leq p \leq 40$$

where err_i is the log-likelihood loss for holdout sample i

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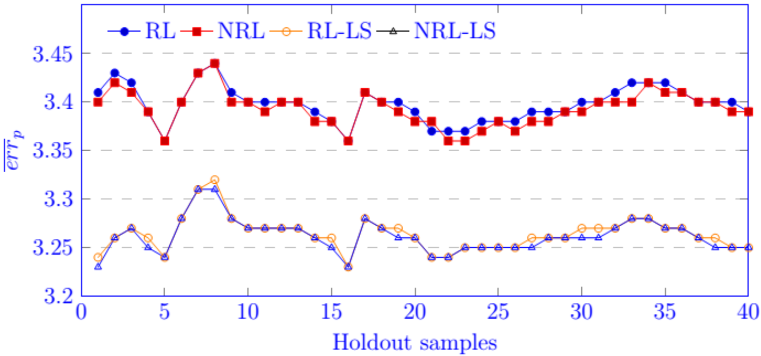
CROSS VALIDATION



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BEYOND THE RECURSIVE LOGIT...

- Following Aguirregabiria and Mira (2010)
- Individuals make sequential decisions, are forward looking and maximize expected intertemporal utility
- Parameters to be estimated describe decision makers preferences and beliefs about the future

INTRODUCTION

- We assume that time is discretized $t = 1, 2, \ldots, T$ and the horizon T can be finite of infinite
- A vector of state variables s_{nt} for individual n and time t describes all relevant information for the decision of interest known at t
- n takes an action/choice $a_{nt} \in A_{nt}$

INTRODUCTION

- Instantaneous/current utility of action a_{nt} is $u(a_{nt}, s_{nt}; \beta)$ (time separability is assumed)
- An individual's preferences over time are then represented by

$$\sum_{j=0}^{T} \rho^j u(a_{n,t+j}, s_{n,t+j}; \beta)$$

where $\rho \in (0,1)$ is a discount factor (ρ is assumed constant over time)

INTRODUCTION

- The decision at t affects the values of future state variables and these values are uncertain to the individual
- The evolution of future states is represented by a Markov transition distribution function $F(s_{n,t+1}|a_{nt}, s_{nt})$

INTRODUCTION

- At each time t, an individual observes s_{nt} and chooses $a_{nt} \in A_{nt}$ to maximize expected utility

$$E\left(\sum_{j=0}^{T-t} \rho^j u(a_{n,t+j}, s_{n,t+j}; \beta) | a_{nt}, s_{nt}\right)$$

INTRODUCTION

- Value function recursively defined by Bellman's equation

$$V(s_{nt}) = \max_{a \in A_{nt}} \left[u(a, s_{nt}; \beta) + \rho \int V(s_{n,t+1}) dF(s_{n,t+1} | a, s_{nt}) \right]$$

- Parameters to be estimated can be included in the instantaneous utilities and the transition probabilities in addition to the discount factor

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DATA

- Panel data of $n = 1, \ldots, N$ individuals
- Actions (choice variable) a_{nt} are observed
- $s_{nt} = (x_{nt}, \varepsilon_{nt})$ where attributes in x_{nt} are observed
- Data $\{a_{nt}, x_{nt} : n = 1, \ldots, N; t = 1, \ldots, T_n\}$

RUST'S MODEL

- Rust (1987) formulated a dynamic discrete choice model which has several similarities with a static logit
- We present these assumptions informally (see e.g. Aguirregabiria and Mira, 2010, for details)

RUST'S MODEL

- Instantaneous utilities are additively separable in observed and unobserved components $u(a, s_{nt}) = v(a, x_{nt}) + \varepsilon_{nt}(a)$
- Random terms $\varepsilon_{nt}(a)$ are i.i.d. Extreme value type I with zero mean and are independent over time, individuals and actions and are also independent of everything else in the model
- $x_{it} \in X$ is discrete and finite
- These assumptions imply that $F(s_{n,t+1}|a_{nt}, s_{nt}) = F_{\varepsilon}(\varepsilon_{n,t+1})F_x(x_{n,t+1}|a_{nt}, x_{nt})$

RUST'S MODEL

The "integrated value function" $\bar{V}(x_{nt}) \equiv \int V(x_{nt}, \varepsilon_{nt}) dF(\varepsilon_{nt})$ is the unique solution to the "integrated Bellman equation"

$$\bar{V}(x_{nt}) = \int \max_{a \in A_{nt}} \left[v(a, x_{nt}; \beta) + \varepsilon_{nt}(a) + \rho \sum_{x_{n,t+1}} \bar{V}(x_{n,t+1}) f_x(x_{n,t+1} | a, x_{nt}) \right] dF(\varepsilon_{nt})$$

To simplify coming formulae, we define a choice specific value function

$$V(a, x_{nt}; \beta) = v(a, x_{nt}; \beta) + \varepsilon_{nt}(a) + \rho \sum_{x_{n,t+1}} \bar{V}(x_{n,t+1}) f_x(x_{n,t+1} | a, x_{nt})$$

RUST'S MODEL

- We now define the probability of choosing an action at a given state
- Individuals maximize instantaneous and expected future utilities

$$\alpha(x_{nt}, \varepsilon_{nt}) = \arg\max_{a \in A_{nt}} V(a, x_{nt}; \beta)$$

- We can obtain choice probabilities as in the static logit case by integrating out the random terms

RUST'S MODEL

- Probability of choosing action a given observed state variables x_{nt}

$$P(a|x_{nt};\beta) = \frac{\exp[V(a,x_{nt};\beta)]}{\sum_{j\in A_{nt}} \exp[V(j,x_{nt};\beta)]}$$

- Given the logit model, the expected maximum utility is the logsum so we obtain a familiar expression for the integrated value function

$$\bar{V}(x_{nt}) = \ln\left(\sum_{j\in A_{nt}} \exp\left[v(a,x_{nt}) + \rho \sum_{x_{i,t+1}} \bar{V}(x_{n,t+1}) f_x(x_{n,t+1}|a,x_{nt})\right]\right)$$

CONCLUDING REMARKS

- In the case of recursive logit the value functions are easy solve because the next state is deterministically given by the action
- When this is not the case, the value functions can be more difficult and time consuming to compute
- NFXP algorithm may be too expensive but alternatives exist (e.g. Hotz and Miller, 1993, Aguirregabiria and Mira, 2002)
- Another lecture focuses on the estimation problem

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